

# Cartographic Projection Procedures

## Release 4

### Second Interim Report

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## 1 Release 4.3 Updates.

As with the previous Rel. 4 update manual, this represents new changes reflected in the third subrelease. The main reason for not combining this material with updates in the previous report in one document are the changes in typesetting style and the delays that would be caused with changing style in the previous report.

Changes most obvious to users of the program **proj** are the addition of new projections—the total is now about 110. For programmers using the projection library, the main change is in how to limit the list of projections linked into application programs. Additional, internal changes were made to ease maintenance of the system, but they should be transparent to both user and programmer.

**Manual Style.** This update is concerned with only documenting projections. Waffling by the author about what should be included or ignored are beginning to converge to the style presented here. Description of the Pseudocylindrical class of projections that follows is nearly complete and will probably not change greatly in the final documentation. A few of previous Miscellaneous projections and new additions are included as well as a section on the General Oblique projection.

It was also decided to include the formulary as part of documentation for reference by the serious reader and to make an explicit definition of what is considered by the author to be the mathematical definition of each projection in this system.

Any comments as to this new style are appreciated.

**Apologies.** Because automatic typesetting programs do not always make the best choices, there are several undesirable locating of figures relative to text. These can usually be overcome by extra effort by the author, but such manipulations are likely to be destroyed by later, overall document alterations. Thus, little effort was expended at this preliminary stage in “beautifying” the text.



## 2 Pseudocylindrical Projections.

Pseudocylindrical projections are a result of efforts to minimize the distortion of the polar regions of the cylindrical projections by bending the meridians toward the center of the map as a function of longitude while maintaining the cylindrical characteristic of parallel parallels. These projections are almost exclusively used for small scale global displays and, except for the Sinusoidal projection, only derived for a spherical Earth. Because of the basic definition of pseudocylindrical projections, none are conformal, but many are equal area.

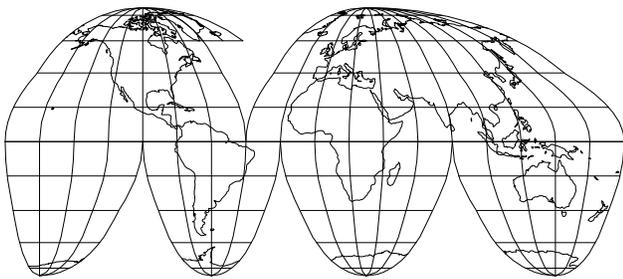


Figure 1: Interrupted Goode Homolosine emphasizing land masses.

To further reduce distortion, pseudocylindrical are often presented in interrupted form that are made by joining several regions with appropriate central meridians and false easting and clipping boundaries. Figs. 1 and 2 show typical construction that are suited for showing respective global land and oceanic regions. To reduce the lateral size of the map, some uses remove an irregular, North-South strip of the mid-Atlantic region so that the western tip of Africa is plotted north of the eastern tip of South America.

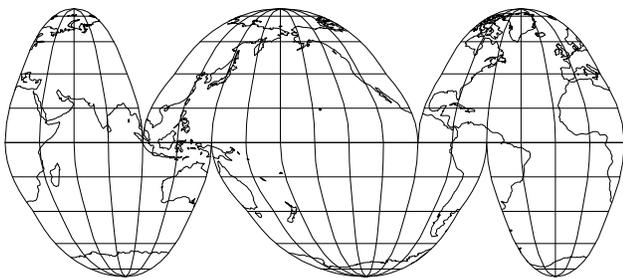


Figure 2: Interrupted Goode Homolosine emphasizing oceanic masses.

Pseudocylindrical are sub-classed into groups based upon the shape of the meridians: sinusoidal, elliptical, parabolic, hyperbolic, rectilinear and miscellaneous. An additional category is based upon whether the meridians come to a point at the pole or are terminated along a straight line—flat-topped.

### 2.0.1 Computations.

A complicating factor in computing the forward projection for pseudocylindricals is that some of the projection formulae use a parametric variable, typically  $\theta$ , which is a function of  $\phi$ . In some cases, the parametric equation is not directly solvable for  $\theta$  and requires use of Newton-Raphson's method of iterative finding the root of  $P(\theta)$ . The defining equations for these cases are thus given in the form of  $P(\theta)$  and its derivative,  $P'(\theta)$ , and an estimating initial value for  $\theta_0 = f(\phi)$ . Refinement of  $\theta$  is made by  $\theta \leftarrow \theta - P(\theta)/P'(\theta)$  until  $|P(\theta)/P'(\theta)|$  is less than predefined tolerance.

When known, formula constant factors are given in rational form (e.g.  $\sqrt{2}/2$ ) rather than a decimal value (0.7071) so that the precision used in the resultant program code constants is determined by the programmer. However, source material may only provide decimal values, typically to 5 or 6 decimal digits. This is adequate in most cases, but has caused problems with the convergence of a Newton-Raphson determination and degrades the determination of numerical derivatives.

Because several of the pseudocylindrical projections have a common computational base, they are grouped into a single module with multiple initializing entry points. This may lead to a minor loss of efficiency, such as adding a zero term in the simple Sinusoidal case of the the Generalized Sinusoidal (2.1.1).

### 2.0.2 Sources.

The principle source for pseudocylindrical formulae is [7]. Many formulae are repeated in Snyder's later works [11] and [10], with the latter adding a few additional projections. Mahling, [2], covers several of the Russian projections but the formulae are often difficult to read. Mahling also has given fourteen pseudocylindrical formulae in [3, Appendix 1] but some discrepancies are found when compared to Snyder's work. For the Robinson Projection (2.6.6), [6] was consulted to verify precision of tabular values and lack of specification of interpolation method. Common pseudocylindricals formulae are also found in Pearson's work: [4] and [5]. Ellipsoid formulae for the Sinusoidal projection is from [9].

## 2.1 Sinusoidal Pseudocylindricals

### 2.1.1 Generalized Sinusoidal

McBryde and Thomas developed a generalized formulas for several of the pseudocylindricals with sinusoidal meridians:

$$\begin{aligned} x &= C\lambda(m + \cos\theta)/(m + 1) \\ y &= C\theta \\ C &= \sqrt{(m + 1)/n} \end{aligned}$$

Table 1: List of pseudocylindrical projections

Projection name	Fig.	Class	Sect.	$H/V$	$P/H$	+proj=	args	file
Boggs Eumorphic	39	A,M	2.6.4	2	0	boggs		boggs.c
Collignon	30	A,R	2.5.1	2	0	collg		collg.c
Craster (Putniņš P <sub>4</sub> )	24	A,P	2.4.1	2	0	crast		crast.c
Denoyer	42	M	2.6.7	2	0.3075	denoy		denoy.c
Eckert I	31	R	2.5.2	2	1/2	eck1		eck1
II	32	A,R	2.5.3	2	1/2	eck2		eck2
III	16	E	2.2.5	2	1/2	eck3		eck3.c
IV	13	A,E	2.2.2	2	1/2	eck4		eck4.c
V	5	S	2.1.3	2	1/2	eck5		eck5.c
VI	3	A,S	2.1.1	2	1/2	eck6		gn_sinu.c
Fahey	43	M	2.6.8	1.4146	0	fahey		fahey.c
Foucaut	36	A,S	2.6.1	1.5708	0	fouc		sts.c
Foucaut Sinusoidal	9	A,M	2.1.7	1.5708	0	fouc_s		fouc_s.c
General Sinusoidal		S	2.1.1			gn_sinu	+n= +m=	gn_sinu.c
Ginsburg VIII	44	M	2.6.9	1.2893	0.5993	gins8		gins8.c
Goode Homolosine	47	A,M	2.6.13	2.3076	0	goode		goode
Hatano	15	A,E	2.2.4	2.0372	1/3	hatano		hatano
Kavraisky VII	19	E	2.2.5	$\sqrt{3}$	1/2	kav7		eck3.c
V	35	A,M	2.6.1	2.0495	0	kav5		sts.c
Loximuthal	45	M	2.6.10			loxim	+lat_1=	loxim.c
McBryde-Thomas								
Sine (No. 1)	34	A,M	2.6.1	2.1192	0	mbt_s		sts.c
Flat-Polar Sine (No. 2)	37	A,M	2.6.2	2.1192	0	mbt_fps		mbt_fps.c
Flat-Polar Sinusoidal (No. 3)	3	A,S	2.1.1	2	1/3	mbtfps		gn_sinu.c
Flat-Polar Quartic (No. 4)	38	A,M	2.6.3	2.2214	1/3	mbtfpq		mbtfpq.c
Flat-Polar Parabolic (No. 5)	29	A,P	2.4.4	2.0944	1/3	mbtftp		mbtftp.c
Mollweide	10	A,E	2.2.1	2	0	moll		moll.c
Putniņš P <sub>1</sub>	17	E	2.2.5	2	0	putp1		eck3.c
P <sub>2</sub>	14	A,E	2.2.3	2	0	putp2		putp2.c
P <sub>3</sub>	27	P	2.4.3	2	0	putp3		putp3.c
P' <sub>3</sub>	28	P	2.4.3	2	1/2	putp3p		putp3.c
P <sub>4</sub>	25	A,P	2.4.2	2	1/2	putp4p		putp4p.c
P <sub>5</sub>	22	H	2.3.2	2	0	putp5		putp5.c
P' <sub>5</sub>	23	H	2.3.2	2	1/2	putp5p		putp5.c
P <sub>6</sub>	20	H,E	2.3.1	2	0	putp6		putp6.c
P' <sub>6</sub>	20	H,E	2.3.1	2	1/2	putp6p		putp6.c
Nell-Hammer	40	A,M	2.6.5	2.7519	1/2	nell_h		nell_h.c
Quartic Authalic	33	A,M	2.6.1	2.2214	0	qua_aut		sts.c
Robinson	41	M	2.6.6	1.9717	0.5322	robin		robin.c
Sinusoidal	3	A,S	2.1.1	2	0	sinu		gn_sinu.c
Urmaev Flat-Polar Sinusoidal		A,S	2.1.2		$\sqrt{1-n^2}$	urmfps	+n=	urmfps.c
V Series		A,M	2.6.12			urm5	+n= +q= +alpha=	urm5.c
Wagner I (Kavraisky VI)	4	A,S	2.1.2	2	1/2	wag1		urmfps.c
II	8	S	2.1.6	2	1/2	wag2		wag2.c
III	7	S	2.1.5			wag3	+lat_ts=	wag3.c
IV (Putniņš P' <sub>2</sub> )	11	A,E	2.2.1	2	1/2	wag4		moll.c
V	12	S	2.2.1	1.9429	0.4531	wag5		moll.c
VI (Putniņš P' <sub>1</sub> )	18	E	2.2.5	2	1/2	wag6		putp1peck3.c
Werenskiold	26	A,H	2.4.2	2	1/2	weren		putp4p.c
Winkel I	46	S	2.1.4	2	1/2	wink1	+lat_ts=	wink1.c
Winkel II	46	M	2.6.11			wink2	+lat_1=	wink2.c

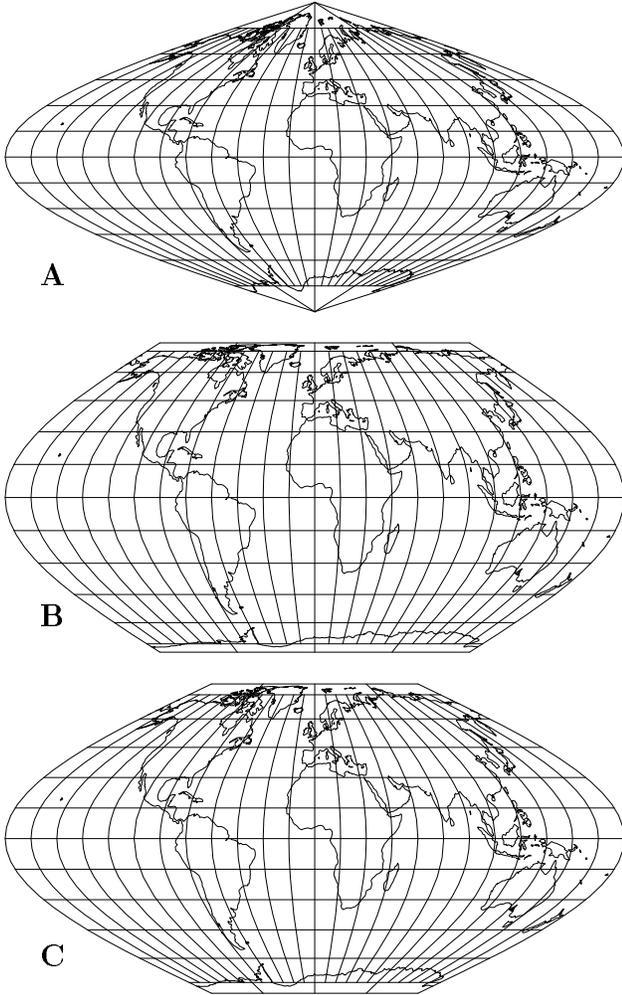


Figure 3: Sinusoidal projections from general formulas: **A**–Sinusoidal, **B**–Eckert VI and **C**–McBryde-Thomas Flat-Polar Sinusoidal.

$$\begin{aligned}
 P(\theta) &= m\theta + \sin \theta - n \sin \phi \\
 P'(\theta) &= m + \cos \theta \\
 \theta_0 &= \phi
 \end{aligned}$$

	$m$	$n$	$C$
Sinusoidal (Sanson-Flamsteed)	0	1	1
Eckert VI	1	$1 + \pi/2$	$2/\sqrt{2 + \pi}$
McBryde-Thomas Flat-Polar Sinusoidal	1/2	$1 + \pi/4$	$\sqrt{6/(4 + \pi)}$

Parameters  $n=n$  and  $m=m$  are required for the general form, `proj=gn_sinu`. The projection is equal-area for all cases.

When  $m = 0$ ,  $P(\theta)$  simplifies and does not need Newton-Raphson iterative solution and in the Sinusoidal case,  $\theta = \phi$ .

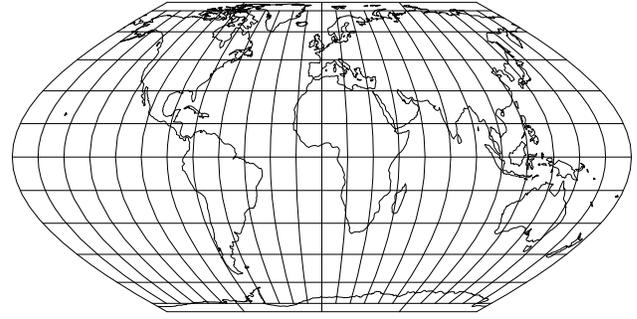


Figure 4: Wagner I.

**Elliptical Earth.** The Sinusoidal projection for the ellipsoidal case becomes:

$$\begin{aligned}
 x &= \lambda \cos \phi (1 - \epsilon^2 \sin^2 \phi)^{-1/2} \\
 y &= M(\phi)
 \end{aligned}$$

The inverse is readily solved by determining  $\phi$  from  $M^{-1}(y)$  and substituting into the  $x$  equation for the solution of  $\lambda$ .

### 2.1.2 Urmaev Flat-Polar Sinusoidal Series

This equal-area system is similar to 2.1.1 where the respective  $x$  and  $y$  axis are multiplied and divided by  $\sqrt{2/3}$  and where  $m = 0$ . The parameter,  $n=n$ , must be specified and is restricted by  $0 < n \leq 1$ . The Wagner I (Kavraysky VI) projection is generated when  $n=\sqrt{3}/2$  or by selecting `proj=wag1`.

$$\begin{aligned}
 x &= (2\sqrt[4]{3}/3)\lambda \cos \theta \\
 y &= 3\theta/(2n\sqrt[4]{3}) \\
 \sin \theta &= n \sin \phi
 \end{aligned}$$

Latitude of true scale on the central meridian is determined by the relation:  $\sin^2 \phi_{ts} = (9 - 4\sqrt{3})/(9 - 4n^2\sqrt{3})$ . The ratio of the length of the poles to the equator is determined by  $\sqrt{1 - n^2}$ .

### 2.1.3 Eckert V

$$\begin{aligned}
 x &= \lambda(1 + \cos \phi)/\sqrt{2 + \pi} \\
 y &= 2\phi/\sqrt{2 + \pi}
 \end{aligned}$$

### 2.1.4 Winkel I

Option `lat_ts= $\phi_{ts}$`  establishes latitude of true scale on central meridian (default =  $0^\circ$  and thus the same as Eckert V). Not equal-area but if  $\cos \phi_{ts} = 2/\pi$  (`lat_ts=50d28'`) the total area of the global map is correct. If  $\phi_{ts} = 0$

$$\begin{aligned}
 x &= \lambda(\cos \phi_{ts} + \cos \phi)/2 \\
 y &= \phi
 \end{aligned}$$

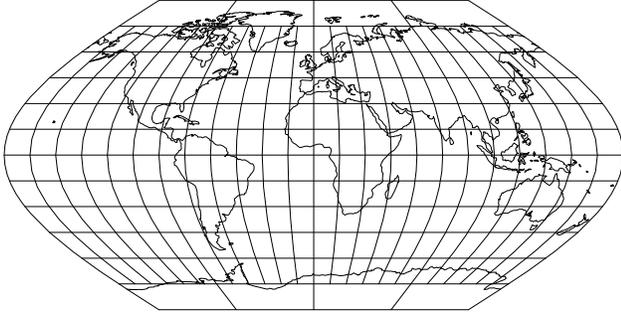


Figure 5: Eckert V.

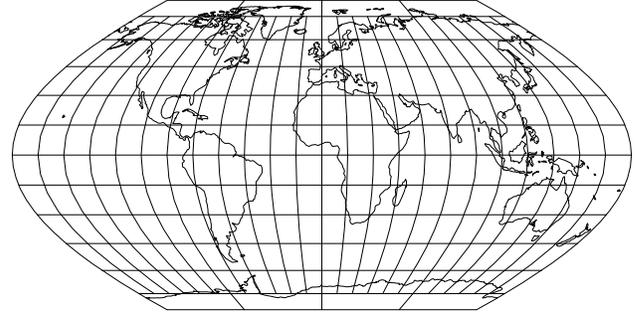
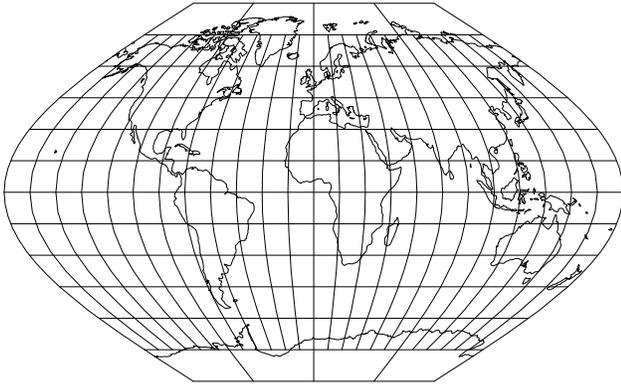
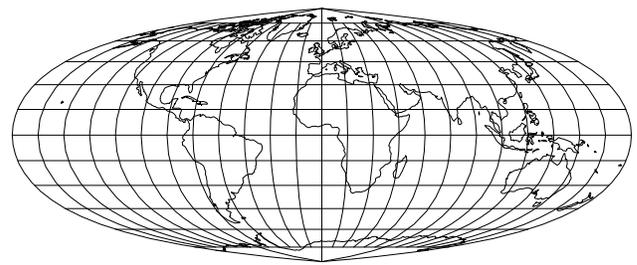


Figure 8: Wagner II.

Figure 6: Winkel I,  $\text{lat\_ts}=50\text{d}28'$ Figure 9: Foucaut Sinusoidal,  $+n=0.5$ .

The  $y$ -axis is based upon a weighted mean of the cylindrical equal-area and the sinusoidal projections. Parameter  $n=n$  is the weighting factor where  $0 \leq n \leq 1$ .

### 2.1.5 Wagner III

$$\begin{aligned} x &= [\cos \phi_{ts} / \cos(2\phi_{ts}/3)] \lambda \cos(2\phi/3) \\ y &= \phi \end{aligned}$$

### 2.1.6 Wagner II

$$\begin{aligned} x &= 0.92483 \lambda \cos \theta \\ y &= 1.38725 \theta \\ \sin \theta &= 0.88022 \sin(0.8855 \phi) \end{aligned}$$

### 2.1.7 Foucaut Sinusoidal.

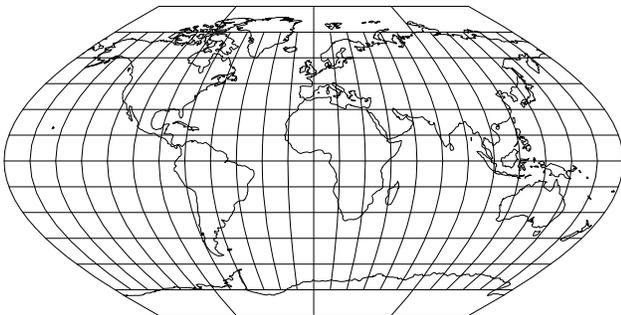


Figure 7: Wagner III.

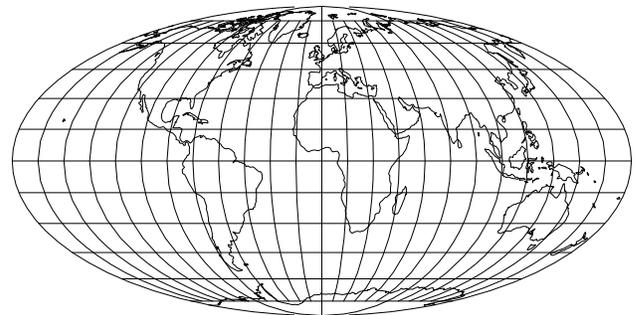


Figure 10: Mollweide.

$$\begin{aligned} x &= \lambda \cos \phi / (n + (1 - n) \cos \phi) \\ y &= n\phi + (1 - n) \sin \phi \end{aligned}$$

For the inverse, the Newton-Raphson method can be used to determine  $\phi$  from the equation for  $y$  above. As  $n \rightarrow 0$  and  $\phi \rightarrow \pi/2$ , convergence is slow but for  $n = 0$ ,  $\phi = \sin^{-1} y$ .

## 2.2 Elliptical Pseudocylindricals.

### 2.2.1 Mollweide, Wagner IV (Putniņš P<sub>2</sub>), and Wagner V

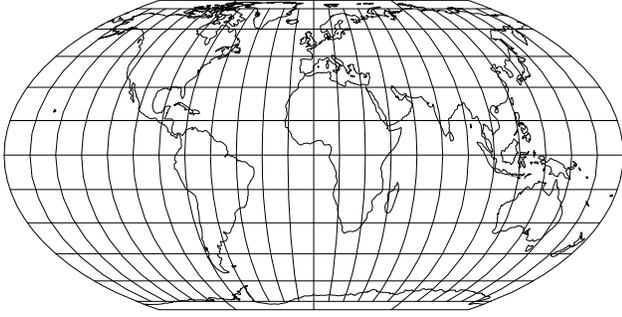


Figure 11: Wagner IV.

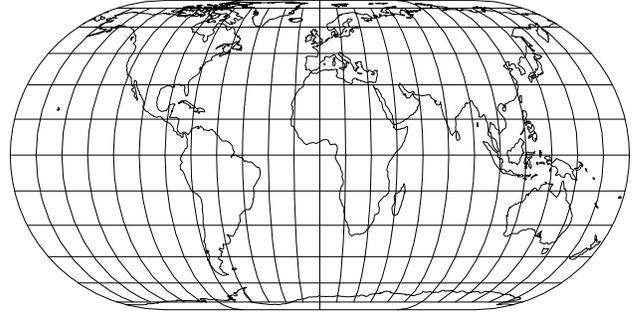


Figure 13: Eckert IV.

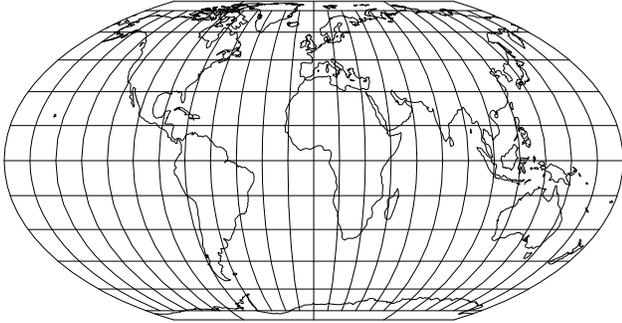


Figure 12: Wagner V

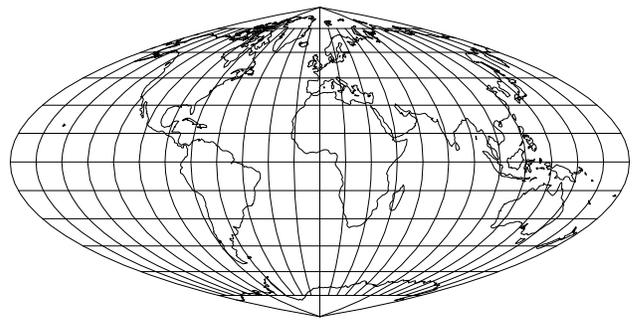


Figure 14: Putniņš P<sub>2</sub>.

Mollweide and Wagner IV are equal area, but Wagner V is not.

$$\begin{aligned}
 x &= C_x \lambda \cos(\theta/2) \\
 y &= C_y \sin(\theta/2) \\
 C_x &= 0.90977 \text{ for Wagner V} \\
 &= 2r/\pi \text{ otherwise} \\
 C_y &= 1.65014 \text{ for Wagner V} \\
 &= r/\sin p \text{ otherwise} \\
 P(\theta) &= \theta + \sin \theta - C_p \sin \phi \\
 C_p &= 3.00896 \text{ for Wagner V} \\
 &= 2p + \sin 2p \text{ otherwise} \\
 P'(\theta) &= 1 + \cos \theta \\
 \theta_0 &= \phi \\
 r &= \sqrt{2\pi \sin p / (2p + \sin 2p)}
 \end{aligned}$$

and where  $p = \pi/2$  for Mollweide and  $p = \pi/3$  for Wagner IV. The parametric equation converges slowly for the Mollweide case.

### 2.2.2 Eckert IV

$$\begin{aligned}
 x &= 2\lambda(1 + \cos \theta) / \sqrt{\pi(4 + \pi)} \\
 y &= 2\sqrt{\pi/(4 + \pi)} \sin \theta \\
 P(\theta) &= \theta + \sin 2\theta + 2 \sin \theta - \frac{(4 + \pi)}{2} \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 &= \theta + \sin \theta(\cos \theta + 2) - \frac{(4 + \pi)}{2} \sin \phi \\
 P'(\theta) &= 2 + 4 \cos 2\theta + 4 \cos \theta \\
 &= 1. + \cos \theta(\cos \theta + 2) - \sin^2 \theta \\
 \theta_0 &= 0.895168\phi + 0.0218849\phi^3 + 0.00826809\phi^5
 \end{aligned}$$

### 2.2.3 Putniņš P<sub>2</sub>

$$\begin{aligned}
 x &= 1.89490\lambda(\cos \theta - 1/2) \\
 y &= 1.71848 \sin \theta \\
 P(\theta) &= 2\theta + \sin 2\theta - 2 \sin \theta - [(4\pi - 3\sqrt{3})/6] \sin \phi \\
 &= \theta + \sin \theta(\cos \theta - 1) - [(4\pi - 3\sqrt{3})/12] \sin \phi \\
 P'(\theta) &= 2 + 2 \cos 2\theta + 2 \cos \theta \\
 &= 1 + \cos \theta(\cos \theta - 1) - \sin^2 \theta \\
 \theta_0 &= 0.615709\phi + 0.00909953\phi^3 + 0.0046292\phi^5
 \end{aligned}$$

The parametric equation converges slowly as  $\phi$  nears  $\pi/2$  and  $\theta$  approaches  $\pi/3$ .

### 2.2.4 Hatano

$$\begin{aligned}
 x &= 0.85\lambda \cos \theta \\
 y &= C_y \sin \theta \\
 P(\theta) &= 2\theta + \sin 2\theta - C_p \sin \phi \\
 P'(\theta) &= 2(1 + \cos \theta) \\
 \theta_0 &= 2\phi
 \end{aligned}$$

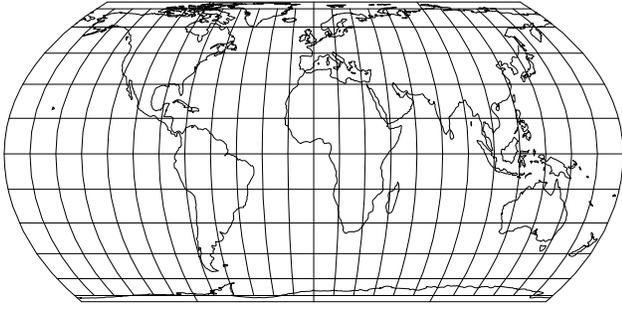


Figure 15: Hatano.

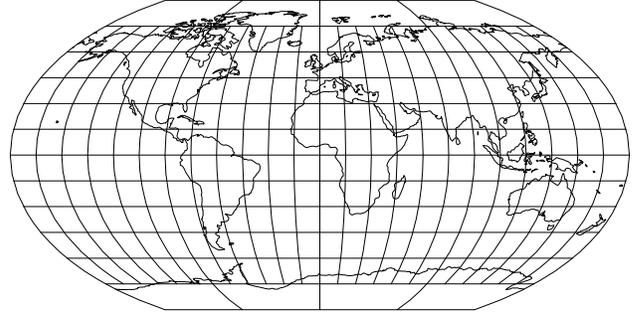


Figure 18: Wagner VI.

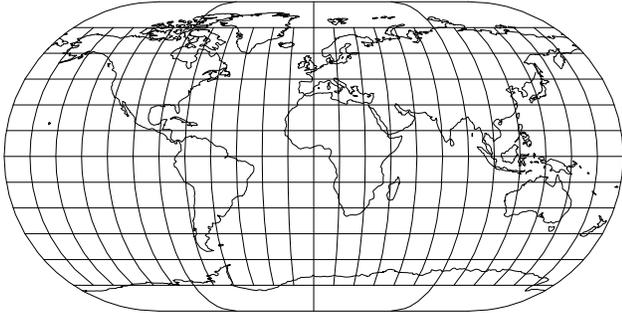


Figure 16: Eckert III.

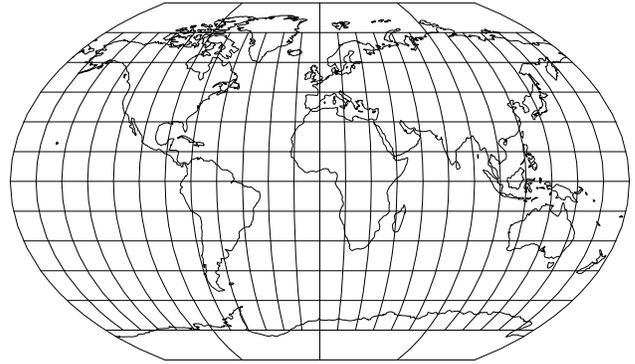


Figure 19: Kavraisky VII.

	$C_y$	$C_p$
$\phi > 0$	1.75859	2.67595
$\phi < 0$	1.93052	2.43763

For  $\phi = 0$ ,  $y \leftarrow 0$  and  $x \leftarrow 0.85\lambda$ .

	$C_x$	$C_y$	$A$	$B$
Putniņš P <sub>1</sub>	0.94745	0.94745	0	3
Wagner VI	1.89490	0.94745	-1/2	3
Eckert III	$\frac{2}{\sqrt{\pi(4+\pi)}}$	$\frac{4}{\sqrt{\pi(4+\pi)}}$	1	4
Kavraisky VII	$\sqrt{3}/2$	1	0	3

**2.2.5 Eckert III, Putniņš P<sub>1</sub>, Wagner VI (Putniņš P'<sub>1</sub>), and Kavraisky VII**

None of these projections are equal-area and are flat-polar when coefficient  $A \neq 0$ .

$$x = C_x \lambda (A + \sqrt{1 - B(\phi/\pi)^2})$$

$$y = C_y \phi$$

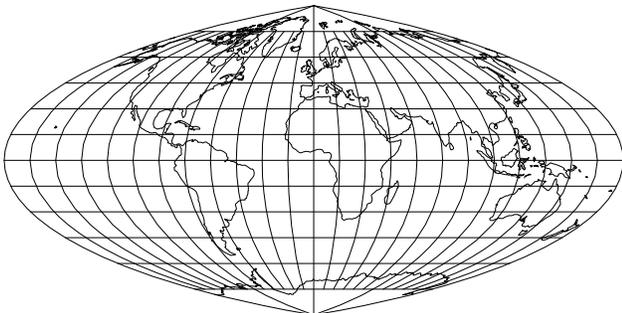


Figure 17: Putniņš P<sub>1</sub>.

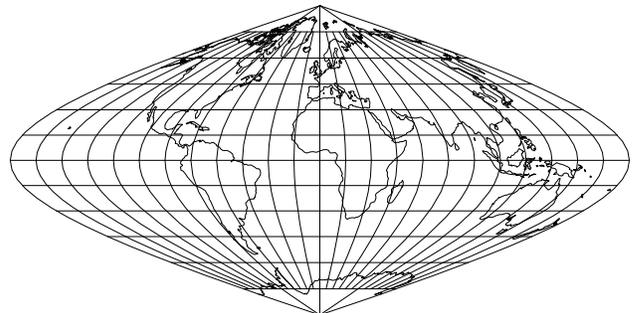


Figure 20: Putniņš P<sub>6</sub>.

**2.3 Hyperbolic Pseudocylindricals**

In this group where the meridians are hyperbolic only four Putniņš forms are given.

**2.3.1 Putniņš P<sub>6</sub> and P'<sub>6</sub>**

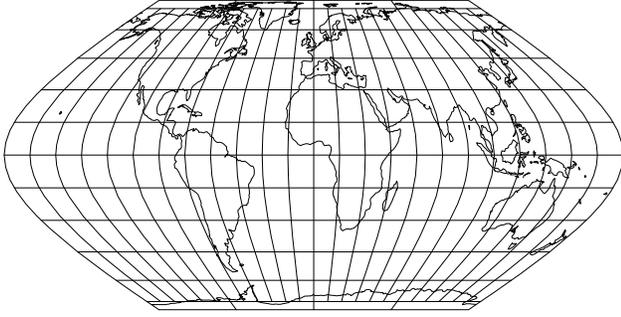


Figure 21: Putniņš P'\_6.

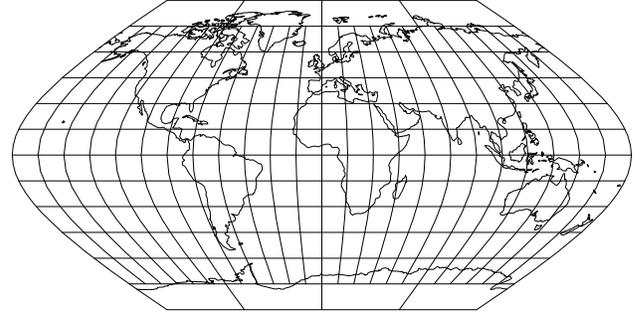


Figure 23: Putniņš P'\_5.

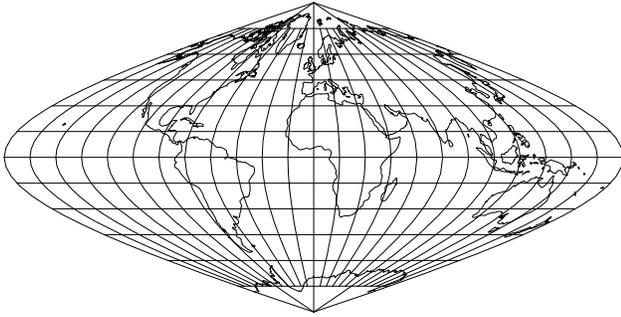


Figure 22: Putniņš P\_5.

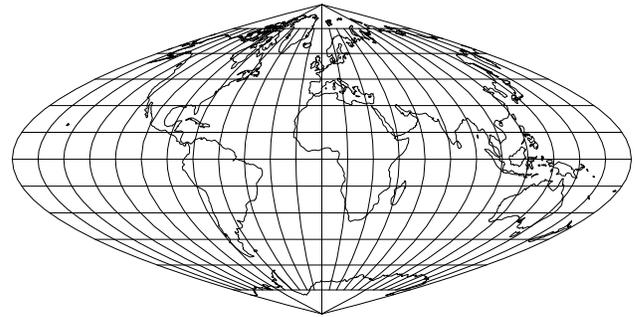


Figure 24: Craster.

Putniņš P\_6 and P'\_6 projections are equal-area with respective pointed and flat poles defined by:

$$\begin{aligned} x &= C_x \lambda (D - (1 + p^2)^{1/2}) \\ y &= C_y p \\ P(p) &= (A - (1 + p^2)^{1/2})p - \ln(p + (1 + p^2)^{1/2}) \\ &\quad - B \sin \phi \\ P'(p) &= A - 2\sqrt{1 + p^2} \\ p_0 &= \phi \end{aligned}$$

where

	P_6	P'_6
$C_x$	1.01346	0.44329
$D$	2	3
$C_y$	0.91910	0.80404
$A$	4.00000	6.00000
$B$	2.14714	5.61125

### 2.3.2 Putniņš P\_5 and P'\_5

Putniņš P\_5 and P'\_5 projections have equally spaced parallels and respectively pointed and flat poles:

$$\begin{aligned} x &= 1.01346 \lambda (A - B \sqrt{1 + 12\phi^2/\pi^2}) \\ y &= 1.01346 \phi \end{aligned}$$

	P_5	P'_5
$A$	2.0	1.5
$B$	1.0	0.5

## 2.4 Parabolic Pseudocylindricals

In this group where the meridians are parabolic.

### 2.4.1 Craster (Putniņš P\_4)

A pointed pole, equal-area projection with standard parallels at 36°46'.

$$\begin{aligned} x &= \sqrt{3/\pi} \lambda [2 \cos(2\phi/3) - 1] \\ y &= \sqrt{3\pi} \sin(\phi/3) \end{aligned}$$

### 2.4.2 Putniņš P'\_4 and Werenskiöld I

This is the flat pole version of Putniņš's P\_4 or Craster's Parabolic:

$$x = C_x \lambda \cos \theta / \cos(\theta/3)$$

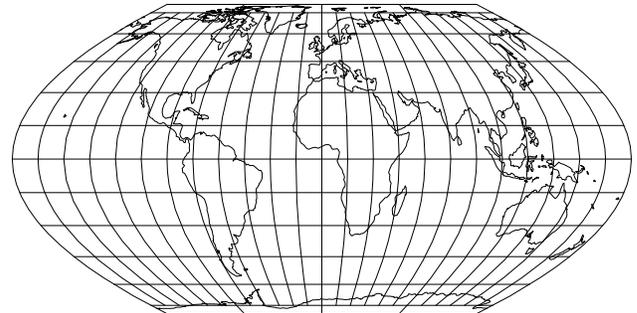


Figure 25: Putniņš P'\_4.

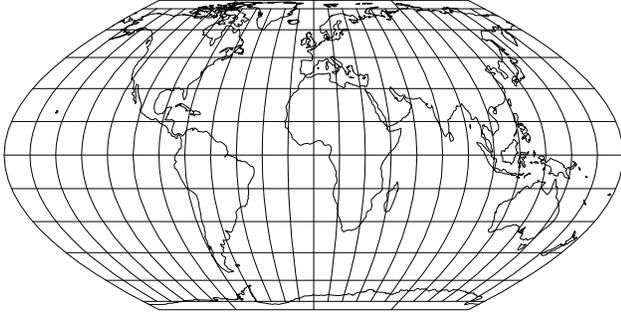
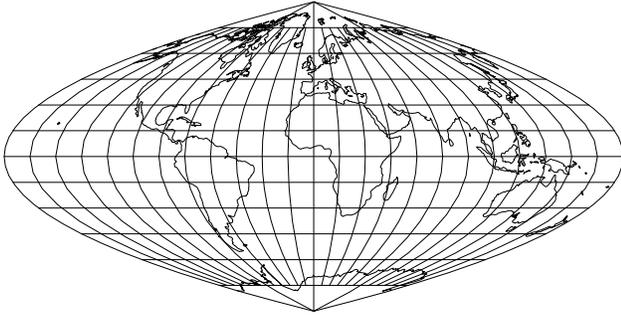


Figure 26: Werenskiold I.

Figure 27: Putniņš  $P_3$ .

$$\begin{aligned} y &= C_y \sin(\theta/3) \\ \sin \theta &= (5\sqrt{2}/8) \sin \phi \end{aligned}$$

where

	$P'_4$	Weren. I
$C_x$	$2\sqrt{0.6/\pi}$	1.0
$C_y$	$2\sqrt{1.2\pi}$	$\pi\sqrt{2}$

#### 2.4.3 Putniņš $P_3$ and $P'_3$

$$\begin{aligned} x &= \sqrt{2/\pi}\lambda(1 - A\phi^2/\pi^2) \\ y &= \sqrt{2/\pi}\phi \end{aligned}$$

where  $A$  is 4 and 2 for respective  $P_3$  and  $P'_3$ .

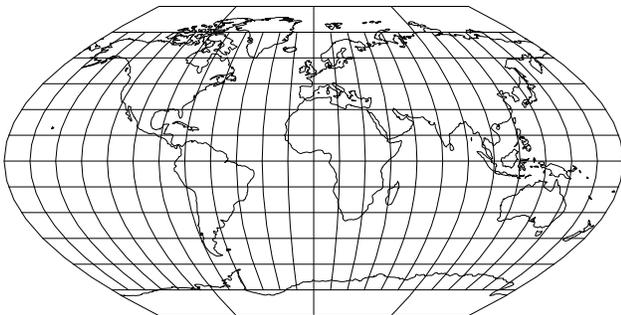
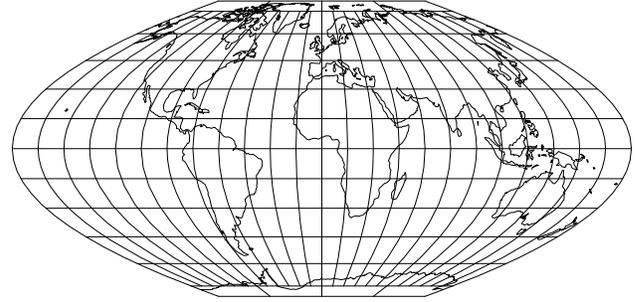
Figure 28: Putniņš  $P'_3$ .

Figure 29: McBryde-Thomas Flat-Polar Parabolic.

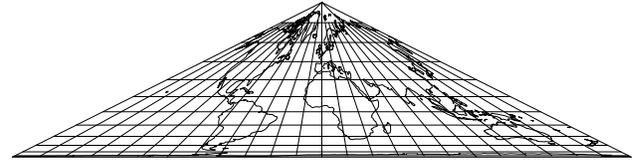


Figure 30: Collignon.

#### 2.4.4 McBryde-Thomas Flat-Polar Parabolic

$$\begin{aligned} x &= \sqrt{6/7}/3\lambda[1 + 2 \cos \theta / \cos(\theta/3)] \\ y &= 3\sqrt{6/7} \sin(\theta/3) \\ P(\theta) &= 1.125 \sin(\theta/3) - \sin^3(\theta/3) - 0.4375 \sin \phi \\ P'(\theta) &= [0.375 - \sin^2(\theta/3)] \cos(\theta/3) \\ \theta_0 &= \phi \end{aligned}$$

## 2.5 Rectilinear

#### 2.5.1 Collignon

$$\begin{aligned} x &= (2/\sqrt{\pi})\lambda\sqrt{1 - \sin \phi} \\ y &= \sqrt{\pi}(1 - \sqrt{1 - \sin \phi}) \end{aligned}$$

#### 2.5.2 Eckert I

$$\begin{aligned} x &= 2\sqrt{2/3}\pi\lambda(1 - |\phi|/\pi) \\ y &= 2\sqrt{2/3}\pi\phi \end{aligned}$$

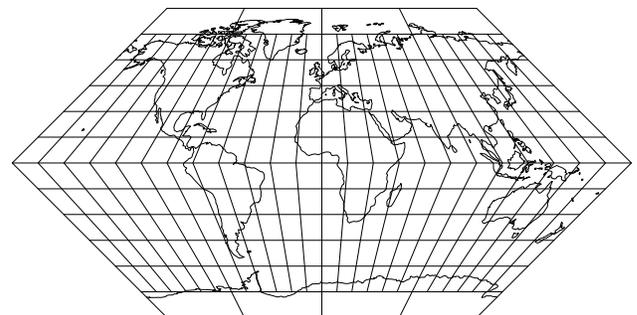


Figure 31: Eckert I.

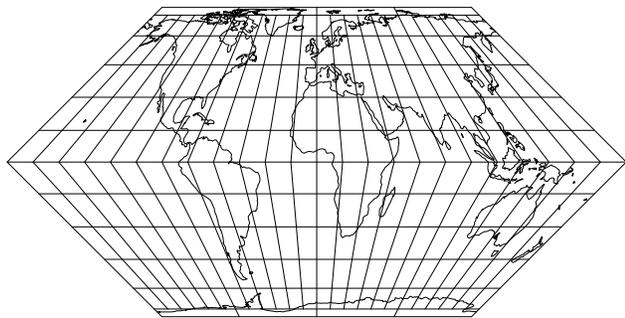


Figure 32: Eckert II.

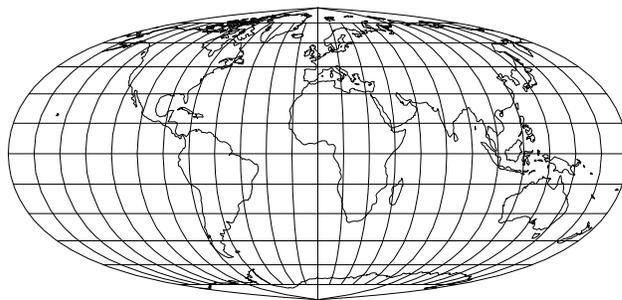


Figure 34: McBryde-Thomas Sine.

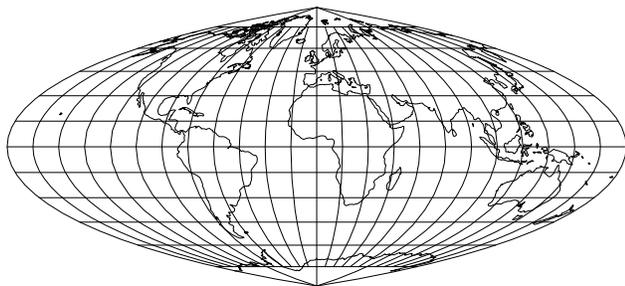


Figure 33: Quartic Authalic.

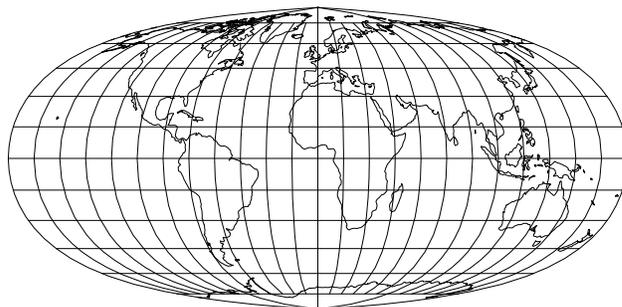


Figure 35: Kavraisky V.

2.5.3 Eckert II

$$x = (2/\sqrt{6\pi})\lambda\sqrt{4 - 3 \sin |\phi|}$$

$$y = \sqrt{2\pi/3}(2 - \sqrt{4 - 3 \sin |\phi|})$$

*y* assumes sign of  $\phi$

2.6 Miscellaneous pseudo/Pseudocylindricals.

2.6.1 Sine-Tangent Series

Sine series:

$$x = (q/p)\lambda \cos \phi / \cos(\phi/q)$$

$$y = p \sin(\phi/q)$$

Tangent series:

$$x = (q/p)\lambda \cos \phi \cos^2(\phi/q)$$

$$y = p \tan(\phi/q)$$

<i>q</i>	<i>p</i>	Sine	Tangent
2	2	Quartic Authalic	Foucaut
1.36509	1.48875	McBryde-Thomas	
1.35439	1.50488	Kavraisky V	

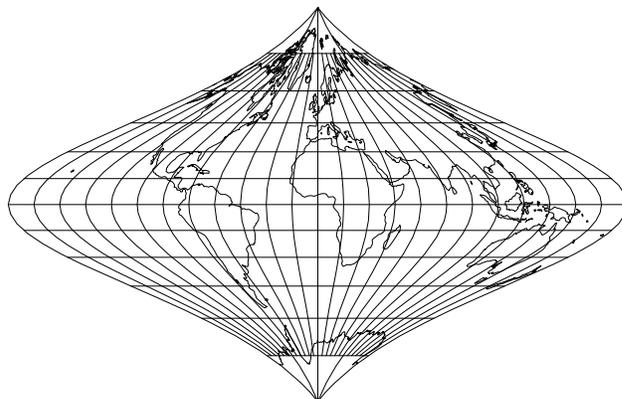


Figure 36: Foucaut.

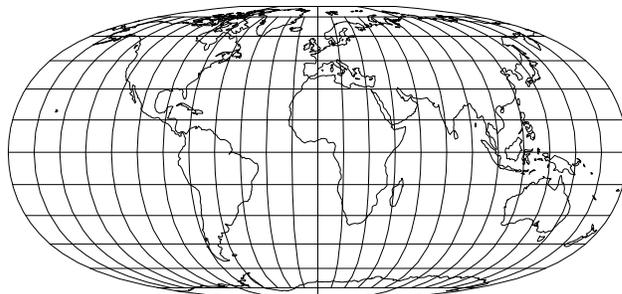


Figure 37: McBryde-Thomas Flat-Polar Sine (No. 1).

2.6.2 McBryde-Thomas Flat-Polar Sine (No. 1).

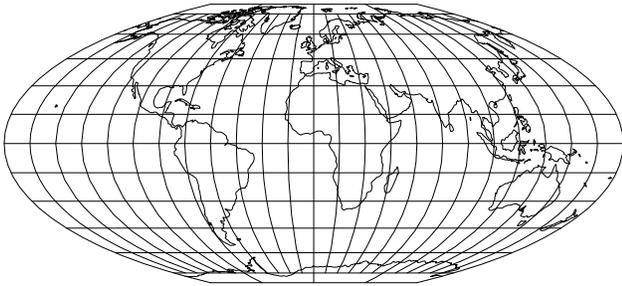


Figure 38: McBryde-Thomas Flat-Polar Quartic.

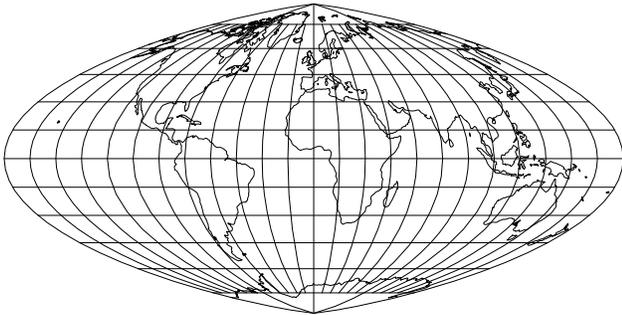


Figure 39: Boggs Eumorphic.

$$\begin{aligned}
 x &= 0.22248\lambda[1 + 3 \cos \theta / \cos(\theta/1.36509)] \\
 y &= 1.44492 \sin(\theta/1.36509) \\
 P(\theta) &= 0.45503 \sin(\theta/1.36509) + \sin \theta - 1.41546 \sin \phi \\
 P'(\theta) &= \frac{0.45503}{1.36509} \cos(\theta/1.36509) + \cos \theta \\
 \theta &= \phi
 \end{aligned}$$

At the moment, there is a discrepancy between formula and claim that 80° parallel length is 1/2 length of equator.

**2.6.3 McBryde-Thomas Flat-Polar Quartic**

$$\begin{aligned}
 x &= \lambda(1 + 2 \cos \theta / \cos(\theta/2))[3\sqrt{2} + 6]^{-1/2} \\
 y &= (2\sqrt{3} \sin(\theta/2)[2 + \sqrt{2}]^{-1/2} \\
 P(\theta) &= \sin(\theta/2) + \sin \theta - (1 + \sqrt{2}/2) \sin \phi \\
 P'(\theta) &= (1/2) \cos(\theta/2) + \cos \theta \\
 \theta &= \phi
 \end{aligned}$$

**2.6.4 Boggs Eumorphic**

$$\begin{aligned}
 x &= 2.00276\lambda(\sec \phi + 1.11072 \sec \theta) \\
 y &= 0.49931(\phi + \sqrt{2} \sin \theta) \\
 P(\theta) &= 2\theta + \sin 2\theta - \pi \sin \phi \\
 P'(\theta) &= 2 + 2 \cos 2\theta \\
 \theta &= \phi
 \end{aligned}$$

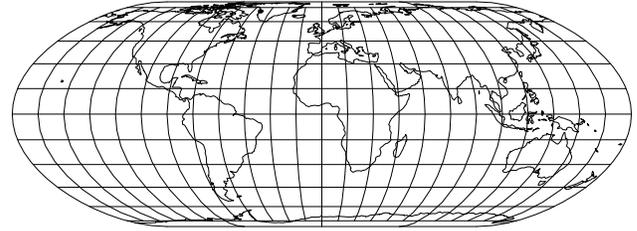


Figure 40: Nell-Hammer.

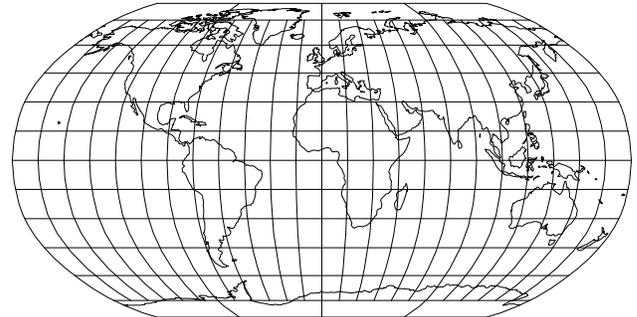


Figure 41: Robinson.

**2.6.5 Nell-Hammer**

$$\begin{aligned}
 x &= \lambda(1 + \cos \phi)/2 \\
 y &= 2(\phi - \tan(\phi/2))
 \end{aligned}$$

**2.6.6 Robinson**

Common for global thematic maps in recent atlases. Not equal-area.

$$\begin{aligned}
 x &= 0.8487\lambda X(|\phi|) \\
 y &= 1.3523Y(|\phi|) \quad y \text{ assumes sign of } \phi
 \end{aligned}$$

where the coefficients of X and Y are determined from the following table:

$\phi^\circ$	Y	X	$\phi^\circ$	Y	X
0	0.0000	1.0000	50	0.6176	0.8679
5	0.0620	0.9986	55	0.6769	0.8350
10	0.1240	0.9954	60	0.7346	0.7986
15	0.1860	0.9900	65	0.7903	0.7597
20	0.2480	0.9822	70	0.8435	0.7186
25	0.3100	0.9730	75	0.8936	0.6732
30	0.3720	0.9600	80	0.9394	0.6213
35	0.4340	0.9427	85	0.9761	0.5722
40	0.4968	0.9216	90	1.0000	0.5322
45	0.5571	0.8962			

Robinson did not define how intermediate values were to be interpolated between the 5° intervals. The **proj** system uses a set of bicubic splines determined for each X-Y set with zero second derivatives at the poles. GCTP [12, program comments] uses Stirling's interpolation with second differences.

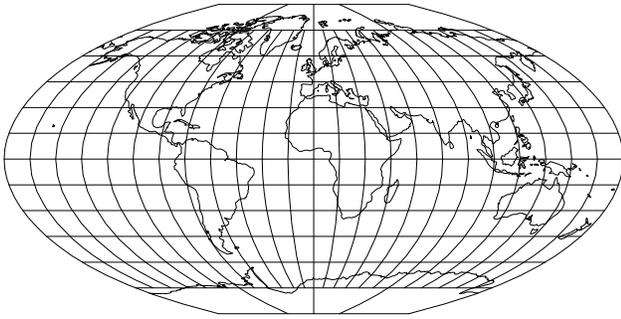


Figure 42: Denoyer.

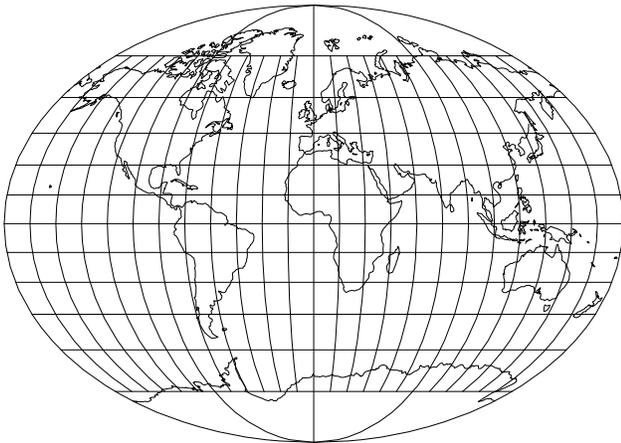


Figure 43: Fahey.

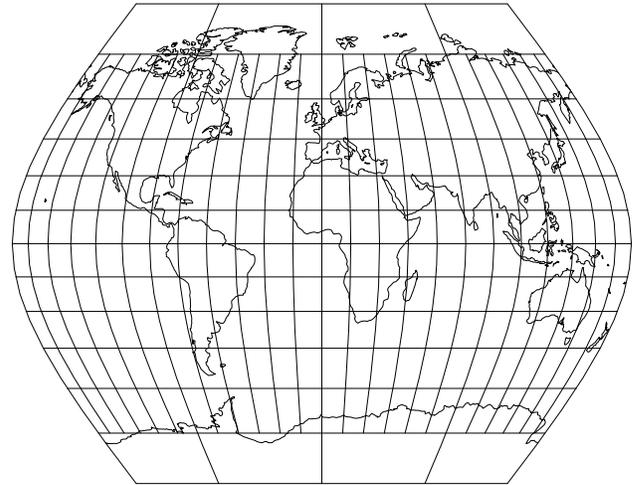


Figure 44: Ginsburg VIII.

### 2.6.7 Denoyer

$$\begin{aligned}x &= \lambda \cos[(0.95 - \lambda/12 + \lambda^3/600)\phi] \\y &= \phi\end{aligned}$$

### 2.6.8 Fahey

$$\begin{aligned}x &= \lambda \cos 35^\circ \sqrt{1 - \tan^2(\phi/2)} \\y &= (1 + \cos 35^\circ) \tan(\phi/2)\end{aligned}$$

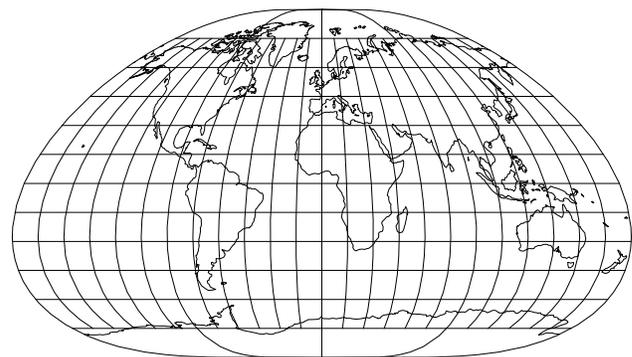
### 2.6.9 Ginsburg VIII or TsNIIGAiK

$$\begin{aligned}x &= \lambda(1 - 0.162388\phi^2)(0.87 - 0.000952426\lambda^4) \\y &= \phi(1 + \phi^2/12)\end{aligned}$$

### 2.6.10 Loximuthal

All straight lines radiating from the point where  $\text{lat}_1=\phi_1$  intersects the central meridian are loxodromes (rhumb lines) and scale along the loxodromes is true.

$$x = \lambda(\phi - \phi_1)/[\ln \tan(\pi/4 + \phi/2) -$$

Figure 45: Loximuthal.  $\text{lat}_1=51\text{d}28$ , Greenwich, England.

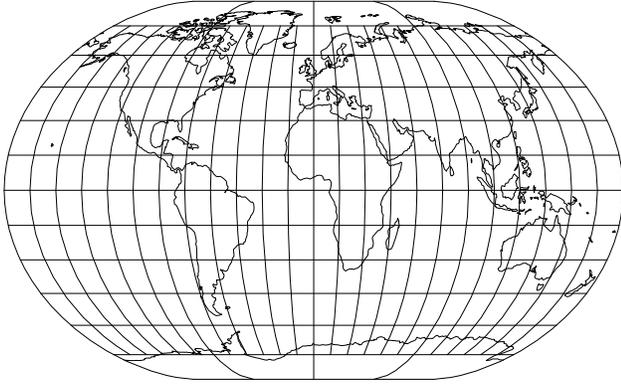


Figure 46: Winkel II,  $\text{+lat}_1=50\text{d}28'$  ( $\cos^{-1}(2/\pi)$ ).

$$\begin{aligned} & \ln \tan(\pi/4 + \phi_1/2)] \text{ for } \phi \neq \phi_1 \\ & = \lambda \cos \phi_1 \text{ for } \phi = \phi_1 \\ y & = \phi - \phi_1 \end{aligned}$$

### 2.6.11 Winkel II

Arithmetic mean of Equirectangular and Mollweide and is not equal-area. Parameter  $\text{lat}_1=\phi_1$  controls standard parallel and width of flat polar extent.

$$\begin{aligned} x & = \lambda(\cos \theta + \cos \phi_1)/2 \\ y & = \pi(\sin \theta + 2\phi/\pi)/4 \\ P(\theta) & = 2\theta + \sin 2\theta - \pi \sin \phi \\ P'(\theta) & = 2 + 2 \cos 2\theta \\ \theta_0 & = 0.9\phi \end{aligned}$$

As with Mollweide,  $P$  converges slowly as  $\phi \rightarrow \pi/2$  and  $\theta \rightarrow \pi/2$ .

### 2.6.12 Urmaev V Series

$$\begin{aligned} x & = m\lambda \cos \theta \\ y & = \theta(1 + q\theta^2/3)/(mn) \\ \sin \theta & = n \sin \phi \\ m & = \cos \alpha / \sqrt{1 - n^2 \sin^2 \alpha} \end{aligned}$$

### 2.6.13 Goode Homolosine

This projection is a combination of the Sinusoidal and Mollweide projections where the Sinusoidal is used for the equatorial regions between the latitudes of  $\pm 40^\circ 44'$  and a corrected Mollweide projection used for the remaining polar regions. The Mollweide correction is to the  $y$  axis with 0.05280 subtracted for northern latitudes and added for southern latitudes. Most often used in the interrupted form (Figs. 1 and 2).

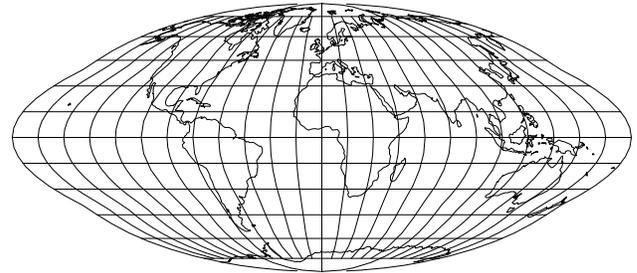


Figure 47: Goode Homolosine.

### 3 Miscellaneous Projections.

Projections that do not clearly fall into previous classifications are placed into the miscellaneous class. This class is further subdivided into subgroupings that are based upon general appearance rather than inherent mathematical or derivative properties.

#### 3.1 Near Pseudocylindricals.

This group of projections are similar to the pseudocylindrical class but with the major exception that they have curved parallels.

##### 3.1.1 Aitoff

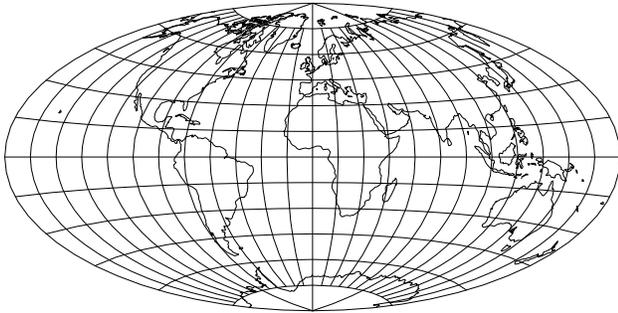


Figure 48: Aitoff

$$\begin{aligned} x &= 2\theta \cos \phi \sin(\lambda/2) / \sin \theta \\ y &= \theta \sin \phi / \sin \theta \\ \cos \theta &= \cos \phi \cos(\lambda/2) \end{aligned}$$

If  $\lambda = \phi = 0$ , then  $x = y = 0$ .

##### 3.1.2 Winkel Tripel

Winkel Tripel is the arithmetic mean of the Aitoff and Equidistant Cylindrical projections with the latter's  $\phi_{ts}$

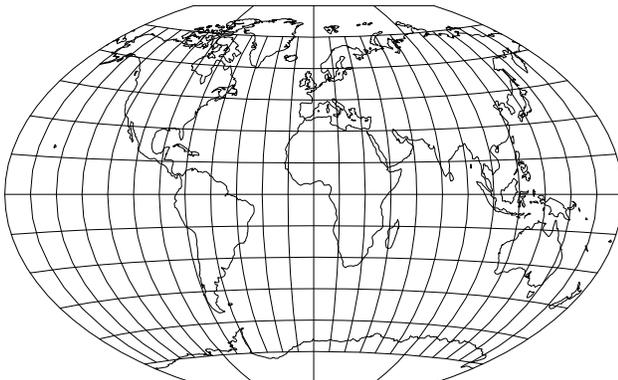


Figure 49: Winkel Tripel, +proj=wintr.

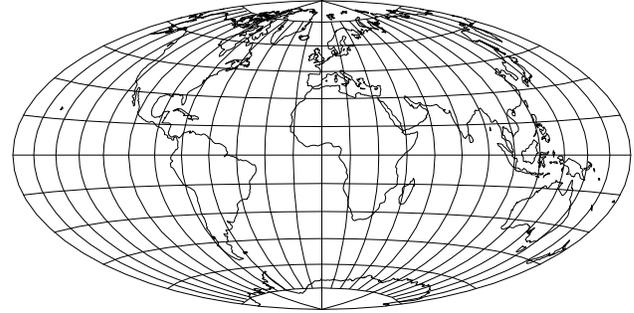


Figure 50: Hammer.

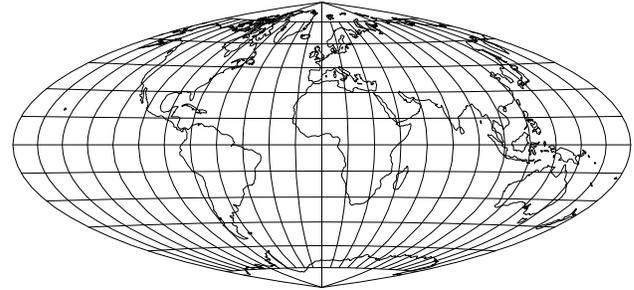


Figure 51: Eckert-Greifendorff, (+proj=hammer +W=0.25).

(latitude of true scale) becoming  $\phi_1$ . If `lat_1= $\phi_1$`  is not specified, Winkel's value of  $\phi_1 = \cos^{-1}(2/\pi)$  or  $50^\circ 27' 35.1945''$  is used. For Bartholomew's variant, use `lat_1=40`.

##### 3.1.3 Hammer (Hammer-Aitoff) and Eckert-Greifendorff.

A popular alternative to pseudocylindricals.

$$\begin{aligned} x &= (\sqrt{2}MD) \cos \phi \sin(W\lambda) \\ y &= (\sqrt{2}D/M) \sin \phi \\ D &= \sqrt{1 + \cos \phi \sin(W\lambda)} \end{aligned}$$

where  $W = 0.5$  for Hammer and  $W = 0.25$  for Eckert-Greifendorff.  $M = 1$  unless overridden with `M=` option which changes the aspect ratio—mainly used for Breisemeister projection (`M= $\sqrt{1.75/2}$` ).

##### 3.1.4 Larrivée.

$$\begin{aligned} x &= \lambda(1 + \cos^{1/2} \phi) / 2 \\ y &= \phi / (\cos(\phi/2) \cos(\lambda/6)) \end{aligned}$$

##### 3.1.5 Wagner VII.

$$\begin{aligned} x &= 2.66723 \cos \theta \sin(\lambda/3) / \cos(\alpha/2) \\ y &= 1.24104 \sin \theta / \cos(\alpha/2) \end{aligned}$$

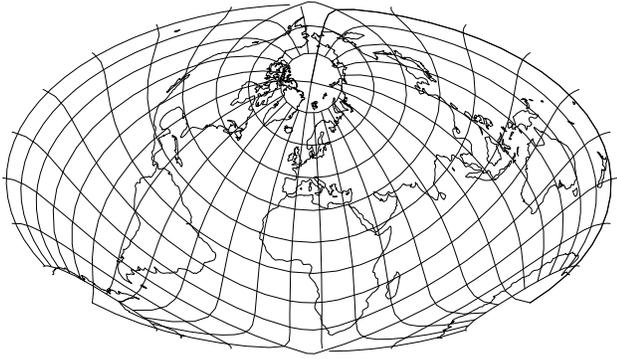


Figure 52: Briesemeister `+proj=ob_tran, +o_proj=hammer, +o_lat_p=45, +o_lon_p=0, +lon_0=10, +M=0.93541`.

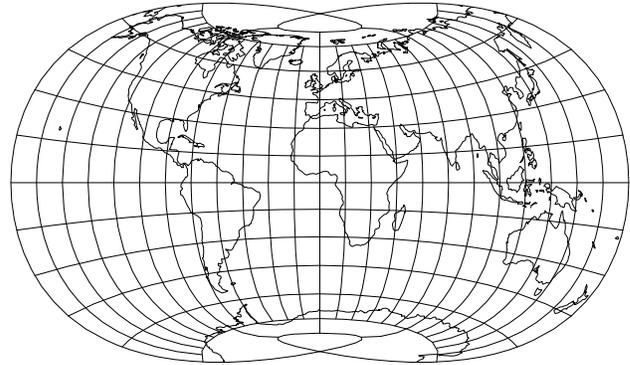


Figure 55: Laskowski, `+proj=lask`.

$$\begin{aligned}\sin \theta &= \sin 65^\circ \sin \phi \\ \cos \alpha &= \cos \theta \cos(\lambda/3)\end{aligned}$$

### 3.1.6 Laskowski.

$$\begin{aligned}x &= \sum_{i=0}^N \sum_{j=0}^M a_{ij} \lambda^i \phi^j \\ y &= \sum_{i=0}^N \sum_{j=0}^M b_{ij} \lambda^i \phi^j\end{aligned}$$

where non-zero coefficients are:

$a_{10}$	0.975534
$a_{12}$	-0.119161
$a_{32}$	-0.0143059
$a_{14}$	-0.0547009
$b_{01}$	1.00384
$b_{21}$	0.0802894
$b_{03}$	0.0998909
$b_{41}$	0.000199025
$b_{23}$	-0.0285500
$b_{05}$	-0.0491032

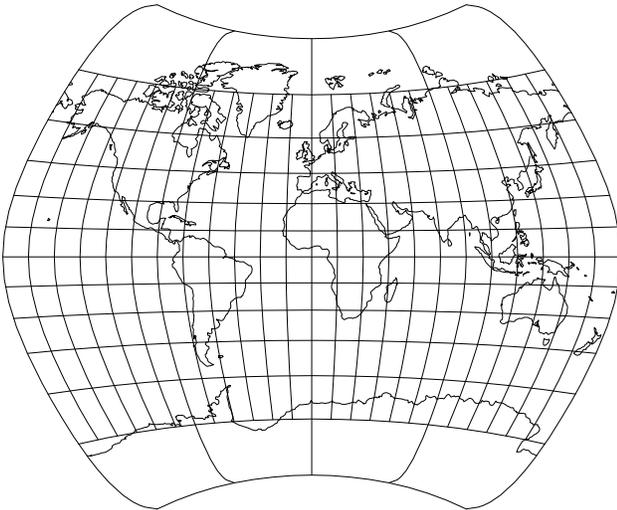


Figure 53: Larrivé, `+proj=larr`.

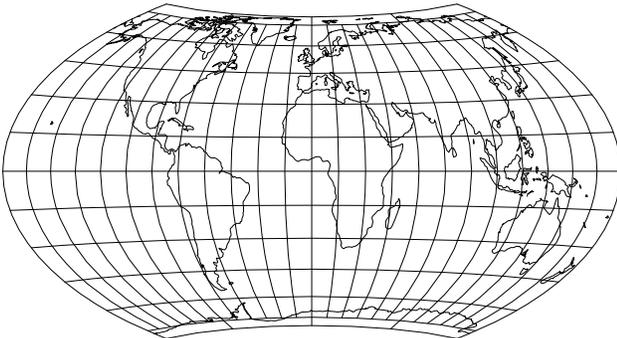


Figure 54: Wagner VII.

## 4 Creating Oblique Projections.

All of the spherical forms of the projections in the **proj** system can be transformed into an *oblique aspect* by making an axis transformation of the geographic coordinates with the following formula:

$$\begin{aligned}\phi' &= \sin^{-1}(\sin \phi_p \sin \phi - \cos \phi_p \cos \phi \cos \lambda) \\ \lambda' &= \lambda + \operatorname{atan2}(\cos \phi \sin \lambda, \\ &\quad \sin \phi_p \cos \phi \cos \lambda + \cos \phi_p \sin \phi)\end{aligned}$$

where  $\lambda_p$  and  $\phi_p$  are the coordinates of the North pole of the transformed coordinate system on the original coordinate system. To use this transformation, the **+o\_proj=name** parameter is used where *name* is the acronym of one of the standard projections—**+o\_proj** is used instead of **+proj**. Parameters **+o\_lat= $\phi_p$**  and **+o\_lon= $\lambda_p$**  are used to set the translated pole position. Any other parameters related to the selected projection *name* are entered as otherwise documented. The parameter **lon\_0** used to shift the central meridian is applied before the transformation in **+ob\_tran** so the effect is to rotate the meridians about the transformed pole and not the pole of the target projection.

To illustrate this procedure, the National Geographic Societies' *Atlas of the World* [1, p. 4] uses the Oblique McBryde-Thomas Flat-Polar projection for a shaded-relief map of the world. Unfortunately, they do not fully annotate the figure (see [8] for comments on this chronic problem) but examination indicates that the transformed pole is at approximately 30° N and 120° W. Fig. 56A shows the overlay of this oblique transformation on the base projection as performed by the options:

```
+o_proj=mbtftpq +o_lat_p=30 +o_lon_b=-120
```

Fig. 56B shows the transformation with coastlines. An element to note is that the 0° meridian of the transformed system follows the  $\lambda_p$  meridian of the untransformed system. Because the creators of the map wanted to emphasize oceanic regions, the axis were rotated by using  $\lambda_0$ . This results in the final options

```
+o_proj=mbtftpq +o_lat_p=30 +o_lon_b=-120 +lon_0=180
```

which results in the map shown in fig. 56C.

Two more examples of transverse pseudocylindrical projections are included here: the Atlantis projection (fig. 57 emphasizes the Atlantic and Arctic Oceans and Close's map (fig. 58 covers the eastern hemisphere. In the latter map, note that the 20° W and 160° E meridians form a circle.

Use of the general oblique transformation is limited to projections assuming a spherical earth. Oblique or transverse projections on an elliptical earth present complex problem that requires specific analysis of each projection and cannot be applied in a general manner.

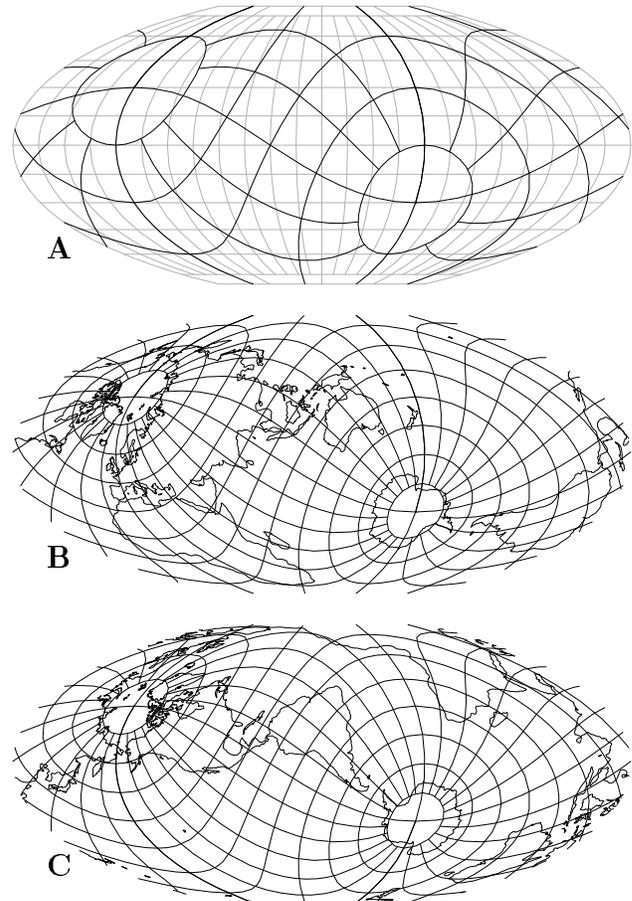


Figure 56: Transverse use of the McBryde-Thomas Flat-Polar Quartic projection: **A**—oblique transformation on base projection, **B**—oblique projection with coastlines and **C**—projection rotated 180° about pole to emphasize oceanic regions.

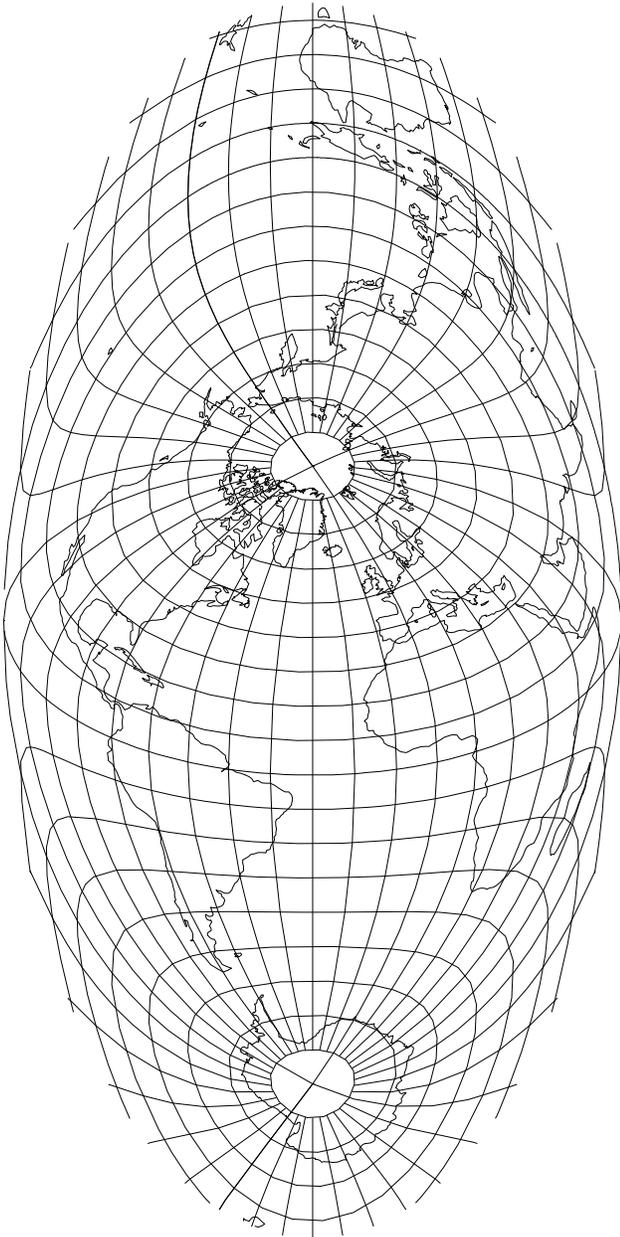


Figure 57: The Atlantis transverse Mollweide projection, `+proj=ob_tran, +o_proj=moll, 10°` graticule.

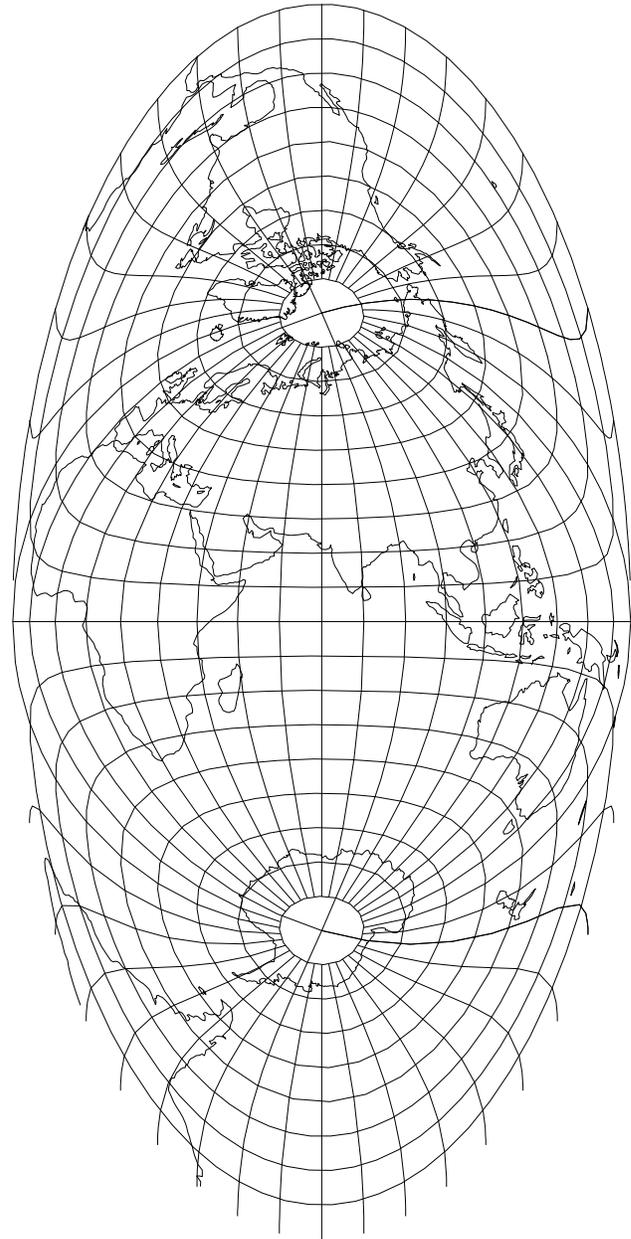


Figure 58: Oblique Mollweide projection proposed by Close, `+proj=ob_tran, +o_proj=moll, +o_lat.p=0, +o_lon.p=90, +lon_0=160. 10°` graticule.

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