Lie theory cheat sheet

Lie group $\mathcal{M}, \circ$		size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$\boldsymbol{\tau}^\wedge \in \mathfrak{m}$	$\boldsymbol{\tau} \in \mathbb{R}^m$
Vector n-D	$\mathbb{R}^n, +$	n	n	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} - \mathbf{v} = \mathbf{0}$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$
Unit Complex number	$\mathrm{S}^1, \cdot$	2	1	$\mathbf{z} \in \mathbb{C}$	$\mathbf{z}^* \mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$
2D Rotation	$\mathrm{SO}(2), \cdot$	4	1	$\mathbf{R}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\theta]_\times = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$
2D Rigid Motion	$\mathrm{SE}(2), \cdot$	9	3	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \in \mathbb{R}^3$
Unit Quaternion	$\mathrm{S}^3, \cdot$	4	3	$\mathbf{q} \in \mathbb{H}$	$\mathbf{q}^* \mathbf{q} = 1$	$\boldsymbol{\theta}/2 \in \mathbb{H}_p$	$\boldsymbol{\theta} \in \mathbb{R}^3$
3D Rotation	$\mathrm{SO}(3), \cdot$	9	3	$\mathbf{R}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\boldsymbol{\theta}]_\times = \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \in \mathfrak{so}(3)$	$\boldsymbol{\theta} \in \mathbb{R}^3$
3D Rigid Motion	$\mathrm{SE}(3), \cdot$	16	6	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6$

Operation	Inverse	Compose	Exp	Log	Right- $\oplus$	Right- $\ominus$
Right Jacobians	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = -\mathbf{Ad}_{\mathcal{X}}$	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\circ\mathcal{Y}} = \mathbf{Ad}_{\mathcal{Y}^{-1}}$ $\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} = \mathbf{I}$	$\mathbf{J}_{\boldsymbol{\tau}}^{\mathrm{Exp}(\boldsymbol{\tau})} = \mathbf{J}_r(\boldsymbol{\tau})$	$\mathbf{J}_{\mathcal{X}}^{\mathrm{Log}(\mathcal{X})} = \mathbf{J}_r^{-1}(\boldsymbol{\tau})$	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplus\boldsymbol{\tau}} = \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})^{-1}}$ $\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X}\oplus\boldsymbol{\tau}} = \mathbf{J}_r(\boldsymbol{\tau})$	$\mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} = -\mathbf{J}_l^{-1}(\boldsymbol{\tau})$ $\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}} = \mathbf{J}_r^{-1}(\boldsymbol{\tau})$

Note: In accordance to `manif` implementation, all Jacobians in this document are **right Jacobians**, whose definition reads: $\frac{\delta f(X)}{\delta X} = \lim_{\varphi \to 0} \frac{f(X \oplus \varphi) \ominus f(X)}{\varphi}$. However, notice that one can relate the left- and right- Jacobians with the Adjoint, ${}^{\mathcal{E}}\frac{\partial f(\mathcal{X})}{\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} {}^{\mathcal{X}}\frac{\partial f(\mathcal{X})}{\partial \mathcal{X}}$, see [1] Eq. (46).

[1] J. Solà, J. Deray, and D. Atchuthan, "A micro Lie theory for state estimation in robotics," Tech. Rep. IRI-TR-18-01, Institut de Robòtica i Informàtica Industrial, Barcelona, 2018. Available at arxiv.org/abs/1812.01537.

$\mathcal{M},\circ$ \ Op	Identity	Inverse	Compose	Act	Exp	Log
$\mathbb{R}^n,+$	$\mathbf{v}=[\mathbf{0}]$	$-\mathbf{v}$	$\mathbf{v}_1+\mathbf{v}_2$	$\mathbf{v}+\mathbf{p}$	$\mathbf{v}$	$\mathbf{v}$
$\mathrm{S}^1,\cdot$	$z=1+i\,0$	z^*	$z_1\ z_2$	$z\ v$	$z=\cos\theta+i\sin\theta$	$\theta=\mathrm{arctan2}(\mathrm{Im}(z),\mathrm{Re}(z))$
$\mathrm{SO}(2),\cdot$	$\mathbf{R}=\mathbf{I}$	$\mathbf{R}^{-1}=\mathbf{R}^\top$	$\mathbf{R}_1\ \mathbf{R}_2$	$\mathbf{R}\cdot\mathbf{v}$	$\mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$	$\theta=\mathrm{arctan2}(r_{21},r_{11})$
$\mathrm{SE}(2),\cdot$	$\mathbf{M}=\begin{bmatrix}\mathbf{R} & \mathbf{t}\\ \mathbf{0} & 1\end{bmatrix}=\mathbf{I}$	$\mathbf{M}^{-1}=\begin{bmatrix}\mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t}\\ \mathbf{0} & 1\end{bmatrix}$	$\mathbf{M}_1\,\mathbf{M}_2=\begin{bmatrix}\mathbf{R}_1\mathbf{R}_2 & \mathbf{t}_1+\mathbf{R}_1\mathbf{t}_2\\ \mathbf{0} & 1\end{bmatrix}$	$\mathbf{M}\cdot\mathbf{p}=\mathbf{t}+\mathbf{R}\mathbf{p}$	$\mathbf{M}=\begin{bmatrix}\mathrm{Exp}(\theta) & \mathbf{V}(\theta)\,\boldsymbol{\rho}\\ \mathbf{0} & 1\end{bmatrix}$ *(1)*	$\boldsymbol{\tau}=\begin{bmatrix}\boldsymbol{\rho}\\ \theta\end{bmatrix}=\begin{bmatrix}\mathbf{V}^{-1}(\theta)\,\mathbf{p}\\ \mathrm{Log}(\mathbf{R})\end{bmatrix}$ *(1)*
$\mathrm{S}^3,\cdot$	$\mathbf{q}=1+i0+j0+k0$	$\mathbf{q}^*=w-ix-jy-jz$	$\mathbf{q}_1\ \mathbf{q}_2$	$\mathbf{q}\,\mathbf{v}\,\mathbf{q}^*$	$\mathbf{q}=\cos\frac{\theta}{2}+\mathbf{u}\sin\frac{\theta}{2}$	$\boldsymbol{\theta}=2\mathbf{v}\frac{\mathrm{arctan2}(\Vert\mathbf{v}\Vert,w)}{\Vert\mathbf{v}\Vert}$
$\mathrm{SO}(3),\cdot$	$\mathbf{R}=\mathbf{I}$	$\mathbf{R}^{-1}=\mathbf{R}^\top$	$\mathbf{R}_1\ \mathbf{R}_2$	$\mathbf{R}\cdot\mathbf{v}$	$\mathbf{R}=\mathbf{I}+\sin\theta\,[\mathbf{u}]_\times+(1-\cos\theta)\,[\mathbf{u}]_\times^2$	$\boldsymbol{\theta}=\frac{\theta(\mathbf{R}-\mathbf{R}^T)^\wedge}{2\sin\theta}$
$\mathrm{SE}(3),\cdot$	$\mathbf{M}=\begin{bmatrix}\mathbf{R} & \mathbf{t}\\ \mathbf{0} & 1\end{bmatrix}=\mathbf{I}$	$\mathbf{M}^{-1}=\begin{bmatrix}\mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t}\\ \mathbf{0} & 1\end{bmatrix}$	$\mathbf{M}_1\,\mathbf{M}_2=\begin{bmatrix}\mathbf{R}_1\mathbf{R}_2 & \mathbf{t}_1+\mathbf{R}_1\mathbf{t}_2\\ \mathbf{0} & 1\end{bmatrix}$	$\mathbf{M}\cdot\mathbf{p}=\mathbf{t}+\mathbf{R}\mathbf{p}$	$\mathbf{M}=\begin{bmatrix}\mathrm{Exp}(\boldsymbol{\theta}) & \mathbf{V}(\boldsymbol{\theta})\,\boldsymbol{\rho}\\ \mathbf{0} & 1\end{bmatrix}$ *(2)*	$\boldsymbol{\tau}=\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix}=\begin{bmatrix}\mathbf{V}^{-1}(\boldsymbol{\theta})\,\mathbf{p}\\ \mathrm{Log}(\mathbf{R})\end{bmatrix}$ *(2)*

$\mathcal{M},\circ$ \ Ad/Jac	$\mathbf{Ad}$	$\mathbf{J}_r$	$\mathbf{J}_l$	$\mathbf{J}^{\mathcal{X}\cdot\mathbf{p}}_{\mathcal{X}}$ (Act)	$\mathbf{J}^{\mathcal{X}\cdot\mathbf{p}}_{\mathbf{p}}$ (Act)
$\mathbb{R}^n,+$	$\mathbf{I}\in\mathbb{R}^{n\times n}$	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$
$\mathrm{S}^1,\cdot$	1	1	1	$\mathbf{R}\,[1]_\times\,\mathbf{v}$	$\mathbf{R}$
$\mathrm{SO}(2),\cdot$	1	1	1	$\mathbf{R}\,[1]_\times\,\mathbf{v}$	$\mathbf{R}$
$\mathrm{SE}(2),\cdot$	$\begin{bmatrix}\mathbf{R} & -[1]_\times\,\mathbf{t}\\ \mathbf{0} & 1\end{bmatrix}$	$\begin{bmatrix}\sin\theta/\theta & (1-\cos\theta)/\theta & (\theta\rho_1-\rho_2+\rho_2\cos\theta-\rho_1\sin\theta)/\theta^2\\ (\cos\theta-1)/\theta & \sin\theta/\theta & (\rho_1+\theta\rho_2-\rho_1\cos\theta-\rho_2\sin\theta)/\theta^2\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}\sin\theta/\theta & (\cos\theta-1)/\theta & (\theta\rho_1+\rho_2-\rho_2\cos\theta-\rho_1\sin\theta)/\theta^2\\ (1-\cos\theta)/\theta & \sin\theta/\theta & (-\rho_1+\theta\rho_2+\rho_1\cos\theta-\rho_2\sin\theta)/\theta^2\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}\mathbf{R} & \mathbf{R}\,[1]_\times\,\mathbf{p}\end{bmatrix}$	$\mathbf{R}$
$\mathrm{S}^3,\cdot$	$\mathbf{R}(\mathbf{q})$	$\mathbf{I}-\frac{1-\cos\theta}{\theta^2}\,[\boldsymbol{\theta}]_\times+\frac{\theta-\sin\theta}{\theta^3}\,[\boldsymbol{\theta}]_\times^2$	$\mathbf{I}+\frac{1-\cos\theta}{\theta^2}\,[\boldsymbol{\theta}]_\times+\frac{\theta-\sin\theta}{\theta^3}\,[\boldsymbol{\theta}]_\times^2$	$-\mathbf{R}(\mathbf{q})\,[\mathbf{v}]_\times$ *(3)*	$\mathbf{R}(\mathbf{q})$ *(3)*
$\mathrm{SO}(3),\cdot$	$\mathbf{R}$	$\mathbf{I}-\frac{1-\cos\theta}{\theta^2}\,[\boldsymbol{\theta}]_\times+\frac{\theta-\sin\theta}{\theta^3}\,[\boldsymbol{\theta}]_\times^2$	$\mathbf{I}+\frac{1-\cos\theta}{\theta^2}\,[\boldsymbol{\theta}]_\times+\frac{\theta-\sin\theta}{\theta^3}\,[\boldsymbol{\theta}]_\times^2$	$-\mathbf{R}\,[\mathbf{v}]_\times$	$\mathbf{R}$
$\mathrm{SE}(3),\cdot$	$\begin{bmatrix}\mathbf{R} & [\mathbf{t}]_\times\,\mathbf{R}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}$	$\begin{bmatrix}\mathbf{J}_r(\boldsymbol{\theta}) & \mathbf{Q}(-\boldsymbol{\rho},-\boldsymbol{\theta})\\ \mathbf{0} & \mathbf{J}_r(\boldsymbol{\theta})\end{bmatrix}$ *(4)*	$\begin{bmatrix}\mathbf{J}_l(\boldsymbol{\theta}) & \mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta})\\ \mathbf{0} & \mathbf{J}_l(\boldsymbol{\theta})\end{bmatrix}$ *(4)*	$\begin{bmatrix}\mathbf{R} & -\mathbf{R}\,[\mathbf{p}]_\times\end{bmatrix}$	$\mathbf{R}$

Some useful identities:

$$\mathcal{X}\oplus\boldsymbol{\tau}=\mathbf{Ad}_{\mathcal{X}}\ \boldsymbol{\tau}\oplus\mathcal{X}\ \mid\ \mathbf{Ad}_{\mathcal{X}}^{-1}=\mathbf{Ad}_{\mathcal{X}^{-1}}\ \mid\ \mathbf{Ad}_{\mathcal{X}\mathcal{Y}}=\mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}}\ \mid\ \mathbf{J}_l(\boldsymbol{\tau})=\mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}\mathbf{J}_r(\boldsymbol{\tau})\ \mid\ \mathbf{J}_l(\boldsymbol{\tau})=\mathbf{J}_r(-\boldsymbol{\tau})$$

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(1) $\mathbf{V}(\theta)=\frac{\sin\theta}{\theta}\mathbf{I}+\frac{1-\cos\theta}{\theta}\,[1]_\times$

(2) $\mathbf{V}(\boldsymbol{\theta})=\mathbf{I}+\frac{1-\cos\theta}{\theta}\,[\mathbf{u}]_\times+\frac{\theta-\sin\theta}{\theta}\,[\mathbf{u}]_\times^2$

(3) $\mathbf{R}(\mathbf{q})=\begin{bmatrix}w^2+x^2-y^2-z^2 & 2(xy-wz) & 2(xz+wy)\\ 2(xy+wz) & w^2-x^2+y^2-z^2 & 2(yz-wx)\\ 2(xz-wy) & 2(yz+wx) & w^2-x^2-y^2+z^2\end{bmatrix}$

(4) $\mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta})=1/2\,[\boldsymbol{\rho}]_\times+\dfrac{\theta-\sin\theta}{\theta^3}([\boldsymbol{\theta}]_\times\,[\boldsymbol{\rho}]_\times+[\boldsymbol{\rho}]_\times\,[\boldsymbol{\theta}]_\times+[\boldsymbol{\theta}]_\times\,[\boldsymbol{\rho}]_\times\,[\boldsymbol{\theta}]_\times)-\dfrac{1-\frac{\theta^2}{2}-\cos\theta}{\theta^4}([\boldsymbol{\theta}]_\times^2\,[\boldsymbol{\rho}]_\times+[\boldsymbol{\rho}]_\times\,[\boldsymbol{\theta}]_\times^2-3\,[\boldsymbol{\theta}]_\times\,[\boldsymbol{\rho}]_\times\,[\boldsymbol{\theta}]_\times)-\dfrac{1}{2}(\dfrac{1-\frac{\theta^2}{2}-\cos\theta}{\theta^4}-3\,\dfrac{\theta-\sin\theta-\frac{\theta^3}{6}}{\theta^5})([\boldsymbol{\theta}]_\times\,[\boldsymbol{\rho}]_\times\,[\boldsymbol{\theta}]_\times^2+[\boldsymbol{\theta}]_\times^2\,[\boldsymbol{\rho}]_\times\,[\boldsymbol{\theta}]_\times)$